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Representation theory of finite semigroups over semirings

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ABSTRACT

We develop the representation theory of a finite semigroup over an arbitrary commutative semiring with unit, in particular classifying the irreducible and minimal representations. The results for an arbitrary semiring are as good as the results for a field. Special attention is paid to the boolean semiring, where we also characterize the simple representations and introduce the beginnings of a character theory.

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1. Introduction

This paper is about the representation theory of finite semigroups over commutative semirings with unit. See [41] for a modern presentation of finite semigroup theory, including a theory of semirings influenced by a semigroup perspective.

Semirings were first introduced by Vandiver in 1934 [49], but did not immediately catch the attention of the mathematical community. The importance of semirings in theoretical computer science was first recognized by Schützenberger [47] in his theory of weighted automata and rational power series [4,45]. Conway also heavily employed semirings in his approach to automata theory [12]. See [33–36] for further applications of semirings to theoretical computer science. Recently, idempotent semirings have entered into mainstream mathematics because they are at the heart of the relatively

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new subject of tropical geometry [13,15,20,43] and tropical algebra [1,21]; a survey can be found in [25]. They also play a role in the notion of characteristic one being developed by Connes et al. [10,11].

Before turning to semirings, we discuss the classical case of fields. Work of Clifford [6,7], Munn [30,31] and Ponizovskii [37] parameterized the irreducible representations of a finite semigroup over a field in terms of the irreducible representations of its maximal subgroups. The reader is referred to [9, Chapter 5] for a full account of this work. Explicit constructions of the irreducible representations were later obtained independently by Rhodes and Zalcstein [42] (which was written in 1968) and by Lallement and Petrich [24] in terms of the Schützenberger representation by monomial matrices [46]; see also [27]. All of these approaches make use of Rees's theorem [38] characterizing 0-simple semigroups up to isomorphism and Wedderburn theory. This viewpoint makes it hard to see how to generalize things to rings, let alone semirings. So it was once believed that we knew everything about representations over the field of complex numbers and nothing about representations over, say, the boolean semiring.

The purpose of this paper is to show that we essentially know just as much about boolean representations as we do about complex ones. In particular, we construct all irreducible representations of a finite semigroup S over any commutative semiring with unit modulo the case of groups. In the case of the boolean semiring we identify the congruence associated to the direct sum of all irreducible representations with one of the congruences from the semilocal theory [41, Chapter 4], generalizing the case of representations over fields [2,39]. In addition to classifying irreducible representations, we also describe all the minimal representations. For the case of boolean representations we are also able to classify the simple representations by showing that they are dual to the minimal ones. In the boolean setting, we propose a notion of characters.

Here we follow the coordinate-free approach from [14], which was used to handle arbitrary commutative rings, in order to deal with semirings. Basic facts about semirings can be found in [16,19,22] or [41, Chapters 8 and 9].

Representing semigroups by boolean matrices is not a new subject. The monoid of $n \times n$ boolean matrices can be identified with the monoid of binary relations on an n -element set. Representations of semigroups by binary relations is an old topic, cf. [5,8,29,32,40,44]. The classical viewpoint is to see binary relations as generalizations of mappings, and hence view the monoid of binary relations as acting on a set with no additive structure. The boolean matrix viewpoint considers an action on a module and so one can speak of invariant subspaces, etc. For instance, the paper [28] studies transitive actions of semigroups by binary relations in the sense that any element of the ground set relates to any other via some semigroup element. For example, any transitive permutation group is a transitive semigroup of binary relations in this sense. But the constant vector will be invariant for any permutation group and so the matrix viewpoint allows us to introduce the more subtle notion of an irreducible module.

2. Preliminaries

We collect here some basic definitions and facts concerning finite semigroups that can be found in any of [9,23,41]. Let S be a (fixed) finite semigroup. If e is an idempotent, then eSe is a monoid with identity e ; its group of units G_e is called the *maximal subgroup* of S at e . Two idempotents e, f are said to be *isomorphic* if there exist $x \in eSf$ and $x' \in fSe$ such that $xx' = e$, $x'x = f$. In this case one can show that eSe is isomorphic to fSf as monoids and hence $G_e \cong G_f$; see Fact 2.1.

If $s \in S$, then $J(s) = S^1sS^1$ is the principal (two-sided) ideal generated by s (here S^1 means S with an adjoined identity). Following Green [17], two elements s, t of a semigroup S are said to be \mathcal{J} -equivalent if $J(s) = J(t)$. In this case one writes $s \mathcal{J} t$. In fact, there is a preorder on S given by $s \leq \mathcal{J} t$ if $J(s) \subseteq J(t)$. This preorder induces an ordering on \mathcal{J} -classes in the usual way.

Fact 2.1. *In a finite semigroup, idempotents e, f are isomorphic if and only if $e \mathcal{J} f$, that is, $SeS = SfS$.*

An element s of a semigroup S is said to be (von Neumann) *regular* if $s = sts$ for some $t \in S$. Each idempotent is of course regular.

Fact 2.2. Let S be a finite semigroup and J a \mathcal{J} -class of S . Then the following are equivalent:

- (1) J contains an idempotent;
- (2) J contains a regular element;
- (3) all elements of J are regular;
- (4) $J^2 \cap J \neq \emptyset$.

A \mathcal{J} -class satisfying the equivalent conditions in Fact 2.2 is called a *regular \mathcal{J} -class*. The poset of regular \mathcal{J} -classes is denoted $\mathcal{U}(S)$. The following standard result from finite semigroup theory will play a role in this paper.

Fact 2.3. Let S be a finite semigroup and J a regular \mathcal{J} -class. Let $e \in J$ be an idempotent. Then $eSe \cap J = G_e$.

Let J be a \mathcal{J} -class of S . Set $J^\star = \{s \in S \mid J \not\subseteq J(s)\}$; it is the ideal of all elements of S that are not \mathcal{J} -above some (equals any) element of J .

The reader is referred to [41, Chapter 9] for basic notions concerning semirings and modules³ over semirings. As the terminology in for semirings and modules is not completely standardized, we say what are our conventions.

A *semiring* [41, Chapter 9] is a 4-tuple $(R, +, \cdot, 0)$ where $(R, +, 0)$ is a commutative monoid, (R, \cdot) is a semigroup, multiplication distributes over addition and 0 is a multiplicative zero, i.e., $0r = 0 = r0$ for all $r \in R$. An *R -module* [41, Chapter 9] is a commutative monoid M equipped with a right action $M \times R \rightarrow M$ satisfying for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$:

- $(m_1 + m_2)r_1 = m_1r_1 + m_2r_1$;
- $m_1(r_1 + r_2) = m_1r_1 + m_1r_2$;
- $m_1(r_1m_2) = (m_1r_1)r_2$;
- $m_10 = 0$;
- $0r_1 = 0$.

If R has a unit, we say that M is a *unitary R -module* if $m1 = m$ for all $m \in M$. Notice that if R is a ring with unit, then any unitary R -module in this sense has additive reduct a group and hence is an R -module in the usual sense of ring theory.

Let us say that a congruence on a module over a semiring is *proper* if it has more than one equivalence class and that it is *trivial* if the associated partition is into singletons. Fix a base commutative semiring ring k with unit.

Definition 2.4. If A is a k -algebra, not necessarily unital, then a right A -module M with $MA \neq 0$ is said to be:

- (1) *simple* if it has only trivial quotient modules;
- (2) *minimal* if it contains no proper non-zero submodules;
- (3) *irreducible* if it is both simple and minimal.

Over a ring k , all three notions coincide. Observe that minimality amounts to asking that, for all $0 \neq m \in M$, the cyclic module mA is M .

The category of (right) A -modules will be denoted $\text{mod-}A$. We adopt here the convention that if A is unital, then by $\text{mod-}A$ we mean the category of unitary A -modules. The reader should verify that all functors considered in this paper respect this convention.

³ We note that some authors use the term *semimodule* for what we call a module.

Definition 2.5 (*Radical and socle*). Let us denote by $\text{rad}(M)$ the intersection of all maximal congruences on a module M ; it is called the *radical* of M . The submodule of M generated by all minimal submodules is denoted $\text{Soc}(M)$ and is called the *socle* of M .

We state and prove Schur's lemma in the semiring context.

Lemma 2.6 (*Schur*). Let M and N be A -modules. Then each non-zero homomorphism $\varphi : M \rightarrow N$ is:

1. injective if M is simple;
2. surjective if N is minimal;
3. an isomorphism if M is simple and N is minimal.

Proof. Suppose that $\varphi \neq 0$. Then $\varphi(M)$ is a non-zero submodule of N and $\ker \varphi$ is a proper congruence on M , from which the result is trivial. \square

The class of minimal modules is closed under quotients.

Proposition 2.7. Let M be a minimal A -module. Then any non-zero quotient module of M is also minimal.

Proof. Suppose that M is minimal and $\varphi : M \rightarrow N$ is a surjective A -module homomorphism. Let N' be a non-zero submodule of N . Then $\varphi^{-1}(N')$ is a non-zero submodule of M and so $M = \varphi^{-1}(N')$, whence $N = N'$. Thus N is minimal. \square

3. Construction of the irreducible and minimal modules

Fix a finite semigroup S and a commutative semiring with unit k . The semigroup algebra kS need not be unital. In this section we construct the irreducible and minimal kS -modules.

3.1. A coordinate-free construction

If M is a kS -module, then $\text{Ann}_S(M) = \{s \in S \mid Ms = 0\}$ is an ideal of S . The following definition, due to Munn [31], is crucial to semigroup representation theory.

Definition 3.1 (*Apex*). A regular \mathcal{J} -class J is said to be the *apex* of a kS -module M if $\text{Ann}_S(M) = J^\uparrow$.

It is easy to see that M has apex J if and only if J is the unique $\leq_{\mathcal{J}}$ -minimal \mathcal{J} -class that does not annihilate M . The notion of apex is closely related to that of lifting \mathcal{J} -classes [41, Lemma 4.6.10]. For further background see [41, Chapter 4.6].

Fix an idempotent transversal $E = \{e_J \mid J \in \mathcal{U}(S)\}$ to the set $\mathcal{U}(S)$ of regular \mathcal{J} -classes and set $G_J = Ge_J$. Let $A_J = kS/kJ^\uparrow$. Notice that the category of kS -modules with apex J can be identified with the full subcategory of $\text{mod-}A_J$ whose objects are modules M such that $Me_J \neq 0$.

Our first goal is to show that every minimal module has an apex. This result is due independently to Munn and Ponizovskii [30,31,37] in the case of fields.

Theorem 3.2. Let M be a minimal kS -module. Then M has an apex.

Proof. Because $MkS \neq 0$, there is a $\leq_{\mathcal{J}}$ -minimal \mathcal{J} -class J such that $J \not\subseteq \text{Ann}_S(M)$. Let $I = S^1JS^1$; of course, I is an ideal of S . Since $I \setminus J$ annihilates M by minimality of J , it follows $0 \neq MkJ = MkI$. From the fact that I is an ideal of S , we may deduce that MkI is a kS -submodule and so by minimality

$$M = MkI = MkJ. \quad (3.1)$$

Therefore, since $JJ^\# \subseteq I \setminus J \subseteq \text{Ann}_S(M)$, it follows from (3.1) that $J^\# = \text{Ann}_S(M)$. Now if J is not regular, then Fact 2.2 implies $J^2 \subseteq I \setminus J$ and hence J annihilates M by (3.1), a contradiction. Thus J is regular and is an apex for M . \square

Now we wish to establish a bijection between irreducible kS -modules with apex J and irreducible kG_J -modules. This relies on a semiring analogue of a well-known result of Green [18, Chapter 6]. Let A be a k -algebra and e an idempotent of A . Then eA is an eAe - A -bimodule and Ae is an A - eAe -bimodule. Hence we have a restriction functor $\text{Res} : \text{mod-}A \rightarrow \text{mod-}eAe$ and induction/coinduction functors $\text{Ind}, \text{Coind} : \text{mod-}eAe \rightarrow \text{mod-}A$ given by

$$\text{Ind}(M) = M \otimes_{eAe} eA, \quad \text{Res}(M) = Me \quad \text{and} \quad \text{Coind}(M) = \text{Hom}_{eAe}(Ae, M).$$

Moreover, Ind is right exact, Res is exact, Coind is left exact and Ind and Coind are the left and right adjoints of Res , respectively (where a functor is left exact if it preserves finite limits and right exact if it preserves finite colimits). This follows from observing that $\text{Hom}_A(eA, M) = Me = M \otimes_A Ae$ and the usual adjunction between hom and the tensor product. Furthermore, it is well known that the unit of the first adjunction gives a natural isomorphism $M \cong \text{Ind}(M)e$ while the counit of the second gives a natural isomorphism $\text{Coind}(M)e \cong M$.

We need the following lemma relating congruences on M and Me .

Proposition 3.3. *Let M be an A -module and let $e \in A$ be an idempotent. Then every congruence R on Me extends to a congruence R' on M by setting $m R' m'$ if and only if $mae R m'ae$ for all $a \in A$. Moreover, R' is the largest congruence whose restriction to Me is contained in R .*

Proof. Clearly R' is k -module congruence. To see that it is an A -module congruence, suppose $b \in A$ and $m R' m'$. Then for any $a \in A$, we have $m(bae) R m'(bae)$ and so $mb R' m'b$ as required. Finally, observe that if $m, m' \in Me$ and $m R' m'$, then $m = mee R m'ee = m'$. Conversely, if $m, m' \in Me$ are R -related and $a \in A$, then $mae = meae R m'ae = m'ae$ and so $m R' m'$. Thus R' extends R .

Finally, suppose that R'' is some other congruence on M whose restriction to Me is contained in R and suppose $m R'' m'$. Then if $a \in A$ one has $mae R m'ae$ and so $m R' m'$ completing the proof. \square

Let M be an A -module and define

$$N(M) = \{(m, m') \mid mae = m'ae, \forall a \in A\}$$

and $L(M) = MeA$. It is easily verified using Proposition 3.3 that $N(M)$ is the largest congruence on M whose restriction to Me is trivial. On the other hand, $L(M)$ is the smallest A -submodule of M with $L(M)e = Me$. The congruence class of 0 under $N(M)$ is

$$K(M) = \{m \in M \mid mae = 0, \forall a \in A\}.$$

It is also the largest submodule of M annihilated by e . The constructions K, L, N are functorial.

Observe that if V is an eAe -module, then

$$L(\text{Ind}(V)) = \text{Ind}(V)eA = V \otimes_{eAe} eAeA = V \otimes_{eAe} eA = \text{Ind}(V). \quad (3.2)$$

Alternatively, one can observe that $\text{Ind}(V)eA$ satisfies the same universal property as $\text{Ind}(V)$.

The analogue of our next result for rings can be found in [18, Chapter 6.2]. The proof for semirings is essentially a modification. We remind the reader that if M is an A -module and N is a submodule, then M/N is the quotient of M by the congruence given by $m \equiv m' \pmod{N}$ if and only if there exist $n, n' \in N$ with $m + n = m' + n'$; it is the largest congruence identifying all elements of N with 0, cf. [41, Chapter 9].

Lemma 3.4. *Let A be a k -algebra and e an idempotent of A .*

- (1) *If M is an irreducible (minimal, simple) A -module, then $Me = 0$ or Me is an irreducible (minimal, simple) eAe -module.*
- (2) *If V is a minimal eAe -module, then the unique maximal A -submodule of $\text{Ind}(V)$ is $K(\text{Ind}(V))$. Moreover, the minimal A -modules M with $Me \cong V$ are (up to isomorphism) the quotients of $\text{Ind}(V)$ by congruences in between congruence modulo $K(\text{Ind}(V))$ and $N(\text{Ind}(V))$.*
- (3) *If V is a simple eAe -module, then $\text{Ind}(V)$ has unique maximal congruence $N(\text{Ind}(V))$. Moreover, if V is irreducible, then the quotient $\text{Ind}(V)/N(\text{Ind}(V))$ is the unique irreducible A -module M with $Me \cong V$.*
- (4) *If V is a minimal eAe -module, then $\text{Coind}(V)$ has unique minimal A -submodule $L(\text{Coind}(V))$. Furthermore, if V is irreducible, then $L(\text{Coind}(V))$ is the unique irreducible A -module M with $Me \cong V$.*

Consequently, restriction yields a bijection between irreducible A -modules that are not annihilated by e and irreducible eAe -modules.

Proof. To prove (1), assume $Me \neq 0$. Suppose first that M is minimal and let $m \in Me$ be non-zero. Then $meA = mA = M$, so $meAe = Me$. Thus Me is minimal. Next suppose that M is simple and let R be a proper congruence on Me . Then R' from Proposition 3.3 is a proper congruence on M and hence trivial. But then R , which is the restriction of R' to Me , is trivial. Thus Me is simple. The irreducible case follows by combining the minimal and simple cases.

Next we turn to (2). Suppose that L is a proper submodule of $\text{Ind}(V)$. Then Le is an eAe -submodule of $\text{Ind}(V)e \cong V$. By minimality of V , either $Le = \{0\}$, and so $L \subseteq K(\text{Ind}(V))$, or $Le = \text{Ind}(V)e$. In the latter case we have $L \supseteq LeA = \text{Ind}(V)eA = \text{Ind}(V)$, where the last equality uses (3.2). Thus $K(\text{Ind}(V))$ is the unique maximal proper submodule. It follows that $\text{Ind}(V)/K(\text{Ind}(V))$ is minimal. Next observe that since $N(\text{Ind}(V))$ has trivial restriction to $\text{Ind}(V)e$, Proposition 2.7 now implies that, for any congruence R between congruence modulo $K(\text{Ind}(V))$ and $N(\text{Ind}(V))$, one has that $\text{Ind}(V)/R$ is minimal and $[\text{Ind}(V)/R]e \cong V$. Finally, suppose that M is a minimal A -module with $Me \cong V$. Then the adjunction yields a non-zero homomorphism $\varphi : \text{Ind}(V) \rightarrow M$ which is surjective by minimality and restricts to an isomorphism of eAe -modules from $\text{Ind}(V)e \rightarrow Me$. In particular, it follows that the congruence $\ker \varphi$ is injective on $\text{Ind}(V)e$ (meaning distinct elements belongs to distinct classes) and hence contained in $N(\text{Ind}(V))$. Also, as $K(\text{Ind}(V))e = 0$ and $Me \neq 0$, minimality implies $\varphi(K(\text{Ind}(V))) = 0$. This establishes (2).

Now we turn to (3). Let V be a simple eAe -module. It is immediate from (3.2) that any proper congruence on $\text{Ind}(V)$ must restrict to a proper congruence on $\text{Ind}(V)e \cong V$. But V is simple and so the only proper congruence on V is the trivial one. Proposition 3.3 now implies that $N(\text{Ind}(V))$ is the unique maximal congruence on $\text{Ind}(V)$. Assume furthermore that V is irreducible. Then $\text{Ind}(V)/N(\text{Ind}(V))$ is minimal, and hence irreducible, and $[\text{Ind}(V)/N(\text{Ind}(V))]e \cong V$ by (2). Uniqueness follows also from (2), since if M is irreducible with $Me \cong V$, then in particular M is minimal and so isomorphic to $\text{Ind}(V)/R$ for a certain congruence $R \subseteq N(\text{Ind}(V))$. But simplicity of M implies that $R = N(\text{Ind}(V))$. This completes the proof of (3).

Finally, to prove (4) first observe that if M is any non-zero A -submodule of $\text{Coind}(V)$, then $Me \neq 0$. Indeed, suppose $Me = 0$ and let $\varphi \in M$. Then, for any x in Ae , we have $\varphi(x) = (\varphi xe)(e) = 0$ since $\varphi xe \in Me = 0$. It follows that $M = 0$. Since $\text{Coind}(V)e \cong V$ is a minimal eAe -module, it now follows that if M is a non-zero A -submodule of $\text{Coind}(V)$, then $Me = \text{Coind}(V)e$ and hence

$$L(\text{Coind}(V)) = \text{Coind}(V)eA \subseteq MeA \subseteq M.$$

This establishes that $L(\text{Coind}(V))$ is the unique minimal A -submodule.

Suppose now that V is irreducible. We must show that any proper congruence on $L(\text{Coind}(V))$ is trivial. Since $\text{Coind}(V)e \cong V$ is irreducible, it follows that any proper congruence on $L(\text{Coind}(V))$ is trivial on $L(\text{Coind}(V))e$ and so by Proposition 3.3 it suffices to show that $N(L(\text{Coind}(V)))$ is the trivial congruence. So suppose $\varphi, \psi \in L(\text{Coind}(V))$ are equivalent and let $a \in A$. Then

$$\varphi(ae) = (\varphi ae)(e) = (\psi ae)(e) = \psi(ae)$$

and hence $\varphi = \psi$. This completes the proof that $L(\text{Coind}(V))$ is irreducible.

Since $L(\text{Coind}(V))e = \text{Coind}(V)eAe = \text{Coind}(V)e \cong V$, it just remains to prove uniqueness. Suppose M is an irreducible A -module with $Me \cong V$. Then the existence of a non-zero element of $\text{Hom}_{eAe}(Me, V) \cong \text{Hom}_A(M, \text{Coind}(V))$ implies that M admits a non-zero homomorphism to $\text{Coind}(V)$. Hence M is isomorphic to an irreducible (and hence minimal) A -submodule of $\text{Coind}(V)$. But $L(\text{Coind}(V))$ is the unique minimal submodule of $\text{Coind}(V)$ and so $M \cong L(\text{Coind}(V))$, as required. \square

We may now establish an analogue of the Clifford–Munn–Ponizovskii theorem for semirings.

Theorem 3.5. *Let S be a finite semigroup, k a commutative semiring with unit and $E = \{e_J \mid J \in \mathcal{U}(S)\}$ be an idempotent transversal to the set $\mathcal{U}(S)$ of regular \mathcal{J} -classes of S . Let G_J be the maximal subgroup G_{e_J} . Define functors $\text{Ind}_J, \text{Coind}_J : \text{mod-}kG_J \rightarrow \text{mod-}kS$ by*

$$\begin{aligned}\text{Ind}_J(V) &= V \otimes_{kG_J} e_J(kS/kJ^{\uparrow}), \\ \text{Coind}_J(V) &= \text{Hom}_{kG_J}((kS/kJ^{\uparrow})e_J, V).\end{aligned}$$

Then:

- (1) *If M is an irreducible (minimal) kS -module with apex J , then Me_J is an irreducible (minimal) kG_J -module.*
- (2) *If V is an irreducible kG_J -module, then*

$$N = \{(u, v) \in \text{Ind}_J(V)^2 \mid uae_J = vae_J, \forall a \in kS/kJ^{\uparrow}\}$$

is the unique maximal congruence on $\text{Ind}_J(V)$ and $\text{Ind}_J(V)/N$ is the unique irreducible kS -module M with apex J such that $Me_J \cong V$.

- (3) *If V is an irreducible kG_J -module, then the unique minimal kS -submodule of $\text{Coind}_J(V)$ is $\text{Coind}_J(V)e_JkS$, which moreover is the unique irreducible kS -module M with apex J such that $Me_J \cong V$.*
- (4) *If V is a minimal kG_J -module, then*

$$K = \{m \in \text{Ind}_J(V) \mid mae_J = 0, \forall a \in kS/kJ^{\uparrow}\}$$

is the unique maximal kS -submodule of $\text{Ind}_J(V)$. Moreover, the minimal kS -modules M with apex J such that $Me_J \cong V$ are up to isomorphism the quotients $\text{Ind}_J(V)/R$ with (retaining the notation of (2)) $R \subseteq N$ and K contained in a single class of R .

Consequently, there is a bijection between the irreducible kS -modules and the irreducible kG_J -modules, where J runs over $\mathcal{U}(S)$.

Proof. Theorem 3.2 implies that every minimal kS -module M has an apex. Again setting $A_J = kS/kJ^{\uparrow}$ for a regular \mathcal{J} -class J , we know that irreducible kS -modules with apex J are in bijection with irreducible A_J -modules M such that $Me_J \neq 0$. It follows directly from Fact 2.3 that

$$e_J A_J e_J = ke_J Se_J / ke_J J^{\uparrow} e_J = kG_J.$$

Lemma 3.4 then yields that irreducible A_J -modules not annihilated by e_J , that is, irreducible kS -modules with apex J , are in bijection with irreducible kG_J -modules in the prescribed manner. Similarly, the minimal kS -modules are as advertised by application of Lemma 3.4. \square

3.2. A construction in coordinates

Let us relate the above construction of the irreducible modules to the explicit ones found in [24,42] for fields. All the facts about finite semigroups used in this discussion can be found in the appendix of [41] or in [23]. According to Green [17], two elements s, t of a semigroup are said to be \mathcal{R} -equivalent if they generate the same principal right ideal. Dually s, t are said to be \mathcal{L} -equivalent if they generate the same principal left ideal.

Once more let S be a finite semigroup, k a commutative semiring with unit and $E = \{e_J \mid J \in \mathcal{U}(S)\}$ an idempotent transversal to the set $\mathcal{U}(S)$ of regular \mathcal{J} -classes of S . Let G_J be the maximal subgroup G_{e_J} . We use L_J and R_J for the \mathcal{L} - and \mathcal{R} -classes of e_J , respectively.

Here we follow [26] to give a concrete description of the irreducible kS -modules. If V is an irreducible kG_J -module, we use \tilde{V} for the corresponding irreducible kS -module. The reader should recall the definition of radical and socle from Definition 2.5.

Proposition 3.6. *Let V be an irreducible kG_J -module. Then there is a natural isomorphism $\text{Hom}_{kS}(\text{Ind}_J(V), \text{Coind}_J(V)) \cong \text{Hom}_{kG_J}(V, V) \neq 0$. Moreover, if $\varphi \in \text{Hom}_{kS}(\text{Ind}_J(V), \text{Coind}_J(V))$ is non-zero, then we have $\ker \varphi = \text{rad}(\text{Ind}_J(V))$ and $\text{Im } \varphi = \tilde{V} = \text{Soc}(\text{Coind}_J(V))$.*

Proof. First note that since $\text{Ind}_J(V), \text{Coind}_J(V)$ are $kS/kJ^{\hat{}}\text{-modules}$, the adjunction yields

$$\text{Hom}_{kG_J}(V, V) = \text{Hom}_{kG_J}(\text{Ind}_J(V)e_J, V) \cong \text{Hom}_{kS}(\text{Ind}_J(V), \text{Coind}_J(V)).$$

Suppose now that $\varphi : \text{Ind}_J(V) \rightarrow \text{Coind}_J(V)$ is a non-zero homomorphism. Because $\text{Ind}_J(V)e_J kS = \text{Ind}_J(V)$ by construction, it follows that

$$\varphi(\text{Ind}_J(V)) = \varphi(\text{Ind}_J(V))e_J kS \subseteq \text{Coind}_J(V)e_J kS = \tilde{V}.$$

Since $\varphi \neq 0$, it follows by irreducibility of \tilde{V} , that $\varphi(\text{Ind}_J(V)) = \tilde{V}$. As $\text{Ind}_J(V)$ has a unique maximal congruence, we conclude that $\ker \varphi = \text{rad}(\text{Ind}_J(V))$. \square

Observe that as k -modules, $kL_J = (kS/kJ^{\hat{}})e_J$ and $kR_J = e_J kS/kJ^{\hat{}}$ by stability. Moreover, the corresponding kG_J - kS -bimodule structure on kR_J is induced by left multiplication by elements of G_J and by the right Schützenberger representation of S on R_J [9,23,41] (i.e., the action of S on R_J by partial functions obtained via restriction of the regular action). To simplify notation, we will use kR_J and kL_J for the rest of this section. Then we have

$$\text{Ind}_J(V) = V \otimes_{kG_J} kR_J,$$

$$\text{Coind}_J(V) = \text{Hom}_{kG_J}(kL_J, V).$$

Multiplication in the semigroup induces a non-zero homomorphism

$$C_J : kR_J \otimes_{kS} kL_J \cong e_J kS/kJ^{\hat{}} \otimes_{kS} (kS/kJ^{\hat{}})e_J \rightarrow e_J (kS/kJ^{\hat{}})e_J \cong kG_J$$

which moreover is a map of kG_J -bimodules.

Let $T \subseteq R_J$ be a complete set of representatives of the \mathcal{L} -classes of J and $T' \subseteq L_J$ be a complete set of representatives of the \mathcal{R} -classes of J . Then G_J acts freely on the left of R_J and T is a transversal for the orbits and dually T' is a transversal for the orbits of the free action of G_J on the right of L_J , see [41, Appendix A]. Thus kR_J is a free left kG_J -module with basis T and kL_J is a free right kG_J -module with basis T' . It is instructive to verify that the associated matrix representation over kG_J of S on kR_J is the classical right Schützenberger representation by row monomial matrices and the representation of S on kL_J is the left Schützenberger representation by column monomial

matrices [9,23,41,42]. Hence if $\ell_J = |T|$ and $r_J = |T'|$, then as kG_J -modules we have $kR_J \cong kG_J^{\ell_J}$ and $kL_J \cong kG_J^{r_J}$. Thus C_J is the bilinear form given by the $\ell_J \times r_J$ -matrix (also denoted C_J) with

$$(C_J)_{ba} = \begin{cases} \lambda_b \rho_a, & \lambda_b \rho_a \in J, \\ 0, & \text{otherwise} \end{cases} \quad (3.3)$$

where $\lambda_b \in T$ represents the \mathcal{L} -class b and $\rho_a \in T'$ represents the \mathcal{R} -class a . Note that $(C_J)_{ba} \in G_J \cup \{0\}$ by stability and C_J is just the usual sandwich (or structure) matrix of the \mathcal{J} -class J coming from the Green–Rees structure theory [9,23,41]. The reader may take (3.3) as the definition of the sandwich matrix if he/she so desires.

Suppose now that V is a kG_J -module. We can consider the induced map

$$V \otimes C_J : V \otimes_{kG_J} kR_J \otimes_{kS} kL_J \rightarrow V \otimes_{kG_J} kG_J \cong V$$

which moreover is non-zero as $v \otimes e_J \otimes e_J \mapsto v$. From the isomorphism

$$\text{Hom}_{kG_J}(V \otimes_{kG_J} kR_J \otimes_{kS} kL_J, V) \cong \text{Hom}_{kS}(V \otimes_{kG_J} kR_J, \text{Hom}_{kG_J}(kL_J, V))$$

we obtain a corresponding non-zero kS -linear map

$$V \otimes C_J : \text{Ind}_J(V) \rightarrow \text{Coind}_J(V)$$

(abusing notation).

Putting together the above discussion with Proposition 3.6, we obtain the following result.

Theorem 3.7. *Let V be an irreducible kG_J -module. Then the irreducible kS -module corresponding to V is the image of the morphism*

$$V \otimes C_J : \text{Ind}_J(V) \rightarrow \text{Coind}_J(V)$$

where C_J is the sandwich matrix for J , i.e., it is the k -span of the rows of $V \otimes C_J$.

Remark 3.8. Note that since kR_J and kL_J are free kG_J -modules with bases T and T' respectively, as k -modules, we have $\text{Ind}_J(V) = V^{\ell_J}$ and $\text{Coind}_J(V) = V^{r_J}$; in particular, these two functors are exact. Moreover, one can easily compute that $V \otimes C_J$ is given via right multiplication by C_J where we view elements of V^{ℓ_J} and V^{r_J} as row vectors with entries in V .

A semigroup S is called *generalized group mapping* if it has a *distinguished* (0-)minimal ideal I on which it acts faithfully on both the left and right [41, Chapter 4]. The distinguished ideal is unique and regular. The following result generalizes a result of Rhodes and Zalcstein for fields [42]. The original proof uses Wedderburn theory, while our proof uses the description of the irreducible modules. Recall that a module M is *faithful* for S if $ms = mt$ for all $m \in M$ implies $s = t$.

Proposition 3.9. *Let S be a finite semigroup and suppose that the irreducible kS -module M is faithful. Then S is generalized group mapping with distinguished ideal J or $J \cup \{0\}$ where J is the apex of M .*

Proof. We just handle the case that S has a zero element 0, as the other case is easier. Also, by Schur's lemma, 0 must act either as the identity (in which case S is trivial) or as the zero endomorphism of M . Suppose that we are in the latter case. By definition of an apex, it is clear that $I = J \cup \{0\}$ is an ideal. Let $e \in E(J)$ and put $V = Me$. Then M is a quotient of $\text{Ind}_J(V)$ and a submodule of $\text{Coind}_J(V)$. It then follows that S acts faithfully on $\text{Ind}_J(V)$ and $\text{Coind}_J(V)$. Now if s, t act the same on the

right of I , then they will act the same on the right of R_e and hence on $\text{Ind}_J(V)$. Thus the action of S on the right of I is faithful. Similarly, if s, t act the same on the left of I , then they act the same on the left of L_e and hence on $\text{Coind}_J(V)$. This completes the proof that S is generalized group mapping. \square

4. The case of idempotent semirings

It turns out that we could have restricted our attention to two cases for irreducible modules: rings (already handled in [14]) and idempotent semirings, as the following observation shows, cf. [50].

Proposition 4.1. *Let S be a finite semigroup and M a simple kS -module. Then the additive structure of M is either an abelian group or a join semilattice with minimum.*

Proof. Define a relation on M by $m \equiv n$ if there exist $j, k \in \mathbb{N}$ so that $jm \in n + M$ and $kn \in m + M$. It is straightforward to verify that \equiv is a congruence. Suppose first that \equiv is trivial and let $m \in M$. Then since $m \equiv m + m$, it follows that $m + m = m$ and so M is a join semilattice with minimum. If \equiv is not proper, then $m \equiv 0$ and hence $0 \in M + m$ and so M is an additive group. \square

Let us now consider the join semilattice case. It turns out that in this case, every representation of a finite group is trivial and so there is exactly one irreducible module over an idempotent semiring associated to any regular \mathcal{J} -class of a finite semigroup.

Proposition 4.2. *Let G be a finite group and k a commutative semiring with unit. Suppose that M is a minimal kG -module with idempotent addition. Then there is a quotient semiring k' of k so that $M = k'$ with trivial action by G .*

Proof. By minimality, it follows that G acts by automorphisms of M . Let $0 \neq m_0 \in M$ and put $m = \sum_{g \in G} m_0 g$. Then m is evidently fixed by G and so the k -span of m is a kG -submodule and hence is M by minimality. If k' is the faithful quotient of k acting on M , then $M = k'$ with the trivial G -action. \square

In particular, if k is a congruence-free commutative idempotent semiring with unit (e.g., the boolean semiring \mathbb{B}), then the trivial action of G on k is the only irreducible kG -module for G a group. Recall that a finite semigroup is *aperiodic* if all its maximal subgroups are trivial.

Corollary 4.3. *Let S be a finite semigroup with a faithful irreducible kS -module M whose addition is idempotent. Then S is generalized group mapping with aperiodic distinguished minimal ideal.*

Proof. By Proposition 3.9, S is generalized group mapping. Let J be the apex of M and let $e \in J$ be an idempotent. Then Me is an irreducible kG_e -module, and hence has trivial action of G_e by Proposition 4.2. By faithfulness, we conclude G_e is trivial. Since G_e is the maximal subgroup of the distinguished ideal of S , this completes the proof. \square

The irreducible $\mathbb{B}S$ -modules admit the following description.

Theorem 4.4. *Let S be a finite semigroup. Then the minimal and irreducible $\mathbb{B}S$ -modules are obtained as follows. Fix a regular \mathcal{J} -class J of S with set A of \mathcal{H} -classes and B of \mathcal{L} -classes. Let $C : B \times A \rightarrow \mathbb{B}$ be the matrix with $C_{ba} = 1$ if and only if the \mathcal{H} -class $a \cap b$ contains an idempotent. Let $\mathbb{B}B$ be the free module on B and consider the natural right action of S on $\mathbb{B}B$. Define a congruence \equiv on $\mathbb{B}B$ by putting $m \equiv n$ if $ms = 0 \Leftrightarrow ns = 0$ for all $s \in S$. Then:*

- (1) The congruence \equiv is the unique maximal proper congruence on $\mathbb{B}B$.
- (2) The module $\mathbb{B}B$ is minimal, as is every proper quotient of $\mathbb{B}B$.
- (3) $\mathbb{B}B/\equiv$ is irreducible and all irreducible modules $\mathbb{B}S$ -modules are of this form for some \mathcal{J} -class.

Alternatively, $\mathbb{B}B/\equiv$ can be identified with the \mathbb{B} -span of the rows of the matrix C .

Proof. First observe that by Proposition 4.2, the only minimal (and hence irreducible) module for a finite group G over \mathbb{B} is \mathbb{B} equipped with the trivial action of G . Thus each \mathcal{J} -class provides a unique irreducible $\mathbb{B}S$ -module M_J , coming from the trivial representation of the maximal subgroup.

Let us observe that C is the tensor product of the structure matrix of J with the trivial representation of the maximal subgroup of G . It follows that M_J can be identified with the \mathbb{B} -span of the rows of C by Theorem 3.7. This theorem also implies that $M_J \cong \mathbb{B}B/\ker C$. Let $m, n \in \mathbb{B}B$. Then first observe that mC is determined by which entries are 0. Now $(mC)_a = \sum_{b \in B} m_b C_{ba}$ and hence is 0 if and only if m is annihilated by the \mathcal{R} -class a . It now follows that $mC = nC$ if and only if, for all $s \in J$, one has $ms = 0 \Leftrightarrow ns = 0$.

Next observe that since J is the apex of M_J , it follows easily by minimality of M_J that if $0 \neq m \in M_J$, then $mJ \neq 0$. Thus if $ms \neq 0$ with $s \in S$, we can find $x \in J$ so that $msx \neq 0$. In particular, it follows by the definition of an apex that $sx \in J$. We conclude that $m \equiv n$ if and only if $ms = 0 \Leftrightarrow ns = 0$ for all $s \in J$. This completes the proof of the (3) of the proposition. The general theory implies that $\mathbb{B}B$ has a unique maximal congruence and hence it must be \equiv .

It remains to prove (2). First notice that by the general theory, the unique maximal submodule of $\mathbb{B}B$ is the congruence class of 0 under the unique maximal congruence. Therefore, the unique maximal submodule of $\mathbb{B}B$ is the set of all vectors annihilated by C . But C is a boolean matrix with no zero rows or columns. Hence, no non-zero vector in $\mathbb{B}B$ is annihilated by C . Thus $\mathbb{B}B$ is minimal and hence so are all its quotients by Proposition 2.7. \square

In [41, Chapter 4.6], it is shown that, for each regular \mathcal{J} -class J of a finite semigroup S , there is a unique congruence \equiv_J on S such that S/\equiv_J is generalized group mapping with aperiodic distinguished ideal and $J^\#$ maps to 0. The resulting quotient is denoted $\text{AGGM}_J(S)$ and the quotient map is written $\Gamma_J : S \rightarrow \text{AGGM}_J(S)$. Consequently, $\text{AGGM}_J(S)$ must then be isomorphic to the image of S under the irreducible representation $S \rightarrow \text{End}_{\mathbb{B}}(M_J)$ constructed in the above proof. In [41, Chapter 4] (see also [48]), it is shown that the intersection over all regular \mathcal{J} -classes J of the congruences \equiv_J is the largest \mathcal{J}' -congruence on S . Recall that a congruence \equiv is a \mathcal{J}' -congruence if $s \equiv t$ and s, t regular implies $s \mathcal{J} t$. Thus we have proved:

Theorem 4.5. *The largest \mathcal{J}' -congruence on a finite semigroup is the congruence associated to the direct sum of all irreducible representations of S over \mathbb{B} .*

The analogous theorems for fields of characteristic 0 and p can be found in [2,39].

A semigroup S is called a *local group* if eSe is a group for each idempotent $e \in S$. The collection of finite local groups is denoted $\mathbb{L}\mathbf{G}$ and is a *pseudovariety*, i.e., is closed under finite products, subsemigroups and homomorphic images. If \mathbf{V} is a pseudovariety, then the Mal'cev product $\mathbb{L}\mathbf{G} \circledast \mathbf{V}$ consists of all finite semigroups S admitting a homomorphism $\varphi : S \rightarrow T$ with $T \in \mathbf{V}$ and $\varphi^{-1}(e) \in \mathbb{L}\mathbf{G}$ for each idempotent e of T . Theorem 4.5 in conjunction with [41, Theorem 4.6.50] yields our next result.

Corollary 4.6. *Let \mathbf{V} be a pseudovariety of semigroups and let S be a finite semigroup. Then $S \in \mathbb{L}\mathbf{G} \circledast \mathbf{V}$ if and only if the image of S under every irreducible representation over \mathbb{B} belongs to \mathbf{V} .*

4.1. Duality and simple boolean modules

If M is a finite join semilattice with identity, then it is automatically a complete lattice. The dual semilattice M^{op} is M with the reverse ordering. It can be identified with the collection of functionals

$f : M \rightarrow \mathbb{B}$ with pointwise operations. Indeed, it is well known [41, Chapter 9] that the functionals are given by choosing $m \in M$ and defining

$$\varphi_m(n) = \begin{cases} 0, & n \leq m, \\ 1, & n \not\leq m \end{cases}$$

and so $m \leq k$ if and only if $\varphi_k \leq \varphi_m$.

From now on we identify M^{op} with the space of functionals. It will be convenient to use the following boolean analogue of the Stone–Weierstrass theorem.

Proposition 4.7. *Let M be a finite join semilattice with identity. Then a subsemilattice with identity $N \subseteq M^{op}$ is equal to M^{op} if and only if it separates the points of M .*

Proof. By the above construction of the functionals, it is clear that M^{op} separates the points of M . Suppose that N is a proper subsemilattice of M with identity and let

$$M' = \{m \mid \varphi_m \in N\}.$$

Then M' is a proper meet-semilattice containing the top of M . Let m be a maximal element of M that does not belong to M' . Then the set of elements strictly above m is non-empty and belongs to M' . It therefore has a meet m' in M' , which must be the unique cover of m . Suppose that $f \in N$. We claim $f(m) = f(m')$. Indeed, if $f = \varphi_k$ with $k \in M'$, then we have two cases. If $m \leq k$, then since $m < k$ we have $m' \leq k$ and so $f(m) = 0 = f(m')$. If $m \not\leq k$, then certainly $m' \not\leq k$ and so $f(m) = 1 = f(m')$. It follows that N does not separate points of M . \square

Suppose now that M is a finite right- $\mathbb{B}S$ module for a finite semigroup S . Then M^{op} is naturally a left $\mathbb{B}S$ -module by putting $sf(m) = f(ms)$.

Theorem 4.8 (Duality). *Let S be a finite semigroup and let M be a finite right $\mathbb{B}S$ -module. Then M is simple (minimal) if and only if M^{op} is minimal (simple).*

Proof. Since $(M^{op})^{op} \cong M$, it suffices to show that M is simple if and only if M^{op} is minimal. Suppose first that M is simple and let N be a non-zero $\mathbb{B}S$ -submodule of M^{op} . Define a congruence \equiv on M by $m \equiv m'$ if $f(m) = f(m')$ for all $f \in N$. This is a congruence because $m \equiv m'$ and $s \in S$ implies that, for all $f \in N$, we have $f(ms) = sf(m) = sf(m') = f(m's)$ as $sf \in N$. Thus $ms \equiv m's$ and so \equiv is a congruence. Since N contains a non-zero functional f , it follows that \equiv is a proper congruence. Thus by simplicity of M , it follows that \equiv is the trivial congruence. Thus the elements of N separate points and so $N = M^{op}$ by Proposition 4.7. It follows that M^{op} is minimal.

Conversely, suppose that M^{op} is minimal and let \equiv be a non-trivial congruence on M . Let $N = \{f \in M^{op} \mid \equiv \subseteq \ker f\}$. Then N is a $\mathbb{B}S$ -submodule of M^{op} . Indeed, trivially $0 \in N$. If $f, g \in N$ and $m \equiv m'$, then $(f + g)(m) = f(m) + g(m) = f(m') + g(m') = (f + g)(m')$. Finally, if $f \in N$, $s \in S$ and $m \equiv m'$, then $ms \equiv m's$ and so $sf(m) = f(ms) = f(m's) = sf(m')$. Thus $sf \in N$. Since M^{op} separates points and \equiv is non-trivial, we conclude that $N \neq M^{op}$ and hence $N = 0$ by minimality. Suppose \equiv is proper. Then $M/\equiv \neq 0$ and has enough functionals to separate points. Hence there is a non-zero functional f on M with $\equiv \subseteq \ker f$, a contradiction. Thus \equiv is not proper. This completes the proof that M is simple. \square

It follows that in principle, we can construct the simple $\mathbb{B}S$ -modules by dualizing the minimal left $\mathbb{B}S$ -modules, which are constructed in Theorem 4.4.

4.2. Characters

In this subsection we propose a definition of the character of a boolean representation and also the notion of a generalized character. These results are preliminary and will be expanded on in a future paper.

Let M be a finite lattice, which we view as a \mathbb{B} -module via its join (and hence we utilize additive notation for the join).

Definition 4.9 (*sjj*). An element $m \in M$ is said to be *strictly join irreducible* (or *sjj*) if $m \neq 0$ and $m = m_1 + m_2$ implies $m = m_1$ or $m = m_2$. See [41, Chapter 6.1.2].

Let us denote by $\min M$ the set of *sjj* elements: it is the unique minimal spanning set of M as a \mathbb{B} -module. In particular, M is a free \mathbb{B} -module if and only if $\min M$ is a basis. Let us say that a decomposition

$$m = \sum_{x \in \min M} c_x x \quad (4.1)$$

is *irredundant* if changing any coefficient c_x from a 1 to a 0 results in a strictly smaller element of M . In other words, an irredundant decomposition is one of the form $m = \sum_{x \in X} x$ with $X \subseteq \min M$ and such that for no proper subset $Y \subsetneq X$ is $m = \sum_{y \in Y} y$. If M is free, then any decomposition of an element m is irredundant and in particular $x \in \min M$ appears in an irredundant decomposition of m if and only if $x \leq m$.

Example 4.10. Let M be the span of $B = \{(1, 0, 0), (1, 1, 0), (0, 1, 1)\}$ in \mathbb{B}^3 . Then $\min M = B$. Now one checks

$$\begin{aligned} (1, 1, 1) &= (1, 0, 0) + (0, 1, 1), \\ (1, 1, 1) &= (1, 1, 0) + (0, 1, 1) \end{aligned}$$

and these are all the irredundant decompositions of $(1, 1, 1)$.

Suppose that S is a finite semigroup and M is a module for S over a field k . Let χ be the character of M . Then if B is a basis for M , one has that

$$\chi(s) = \sum_{b \in B} c_{bs} b$$

where

$$bs = \sum_{b' \in B} c_{bb'} b'.$$

In other words, it sums over all $b \in B$ the multiplicity of b in the decomposition of bs with respect to the basis B (where the multiplicity is taken in the field k). The problem with extending this idea to boolean case is the non-uniqueness of irredundant decompositions. So instead we try and minimize over all decompositions. With this in mind, we proceed to define the character of a $\mathbb{B}S$ -module in two steps.

Definition 4.11 (*Character of a boolean matrix representation*). Suppose that $\varphi : S \rightarrow M_n(\mathbb{B})$ is a matrix representation. Let B be the basis of \mathbb{B}^n . Then define

$$\chi_\varphi(s) = |\{b \in B \mid bs \geq b\}|.$$

Equivalently, $\chi_\varphi(s)$ is the trace $\text{tr}(\varphi(s))$ where we view $\varphi(s)$ as a zero–one matrix over \mathbb{C} . If M is the corresponding $\mathbb{B}S$ -module, then we also use the notation χ_M for its character.

Recall [3] that a boolean representation $\varphi : S \rightarrow M_n(\mathbb{B})$ is *unambiguous* if the product $\varphi(s)\varphi(t)$, viewed as matrices over \mathbb{C} , coincides with $\varphi(st)$. Unambiguous representations play a key role in the theory of rational codes [3]. We shall say that a $\mathbb{B}S$ -module M is *unambiguous* if it is a free \mathbb{B} -module and the corresponding matrix representation is unambiguous. For instance, if S acts on a finite set X , then the $\mathbb{B}S$ -module $\mathbb{B}X$ is unambiguous. From the definition, it is immediate that the character of an unambiguous module is a complex character of the semigroup.

Next suppose that M is a finite $\mathbb{B}S$ -module (with S a finite semigroup). Then there is a natural surjective \mathbb{B} -linear map $\pi : \mathbb{B}(\min M) \rightarrow M$ induced by the identity map on $\min M$. Associated to each set-theoretic section $\sigma : M \rightarrow \mathbb{B}(\min M)$ of π with $\sigma|_{\min M} = 1_{\min M}$ is a $\mathbb{B}S$ -module structure M_σ on $\mathbb{B}(\min M)$ defined by $sx = \sigma(sx)$ for $s \in S$ and $x \in \min M$. Moreover, notice that if S has an identity 1 and M is unitary, then $1x = \sigma(1x) = \sigma(x) = x$ and so M_σ is unitary. In any case, one has that $\pi : M_\sigma \rightarrow M$ is a surjective morphism of $\mathbb{B}S$ -modules.

Definition 4.12 (*Min character*). Let S be a finite semigroup and M a finite $\mathbb{B}S$ -module. Then the *min character* $\chi_M : S \rightarrow \mathbb{N}$ is defined by

$$\chi_M(s) = \min\{\chi_{M_\sigma}(s)\}$$

where σ runs over all set-theoretic sections of $\pi : \mathbb{B}(\min M) \rightarrow M$ with $\sigma|_{\min M} = 1_{\min M}$.

In more concrete terms, to compute $\chi_M(s)$, one first fixes an irredundant decomposition of each element of M . Then one counts how many $x \in \min X$ appear in the chosen irredundant decomposition of xs . Then one minimizes this quantity over all choices made. Of course, if M is a $\mathbb{B}S$ -module that is free as a \mathbb{B} -module, then clearly the two notions of the character of M coincide.

Example 4.13. Let M be the span of $B = \{(1, 0, 0), (1, 1, 0), (0, 1, 1)\}$ in \mathbb{B}^3 . The remaining elements of M are $(0, 0, 0)$ and $(1, 1, 1)$. As observed earlier $\min M = B$. Any admissible section of π must be the identity on all elements of M except $(1, 1, 1)$, which has three lifts:

$$(1, 1, 1) = (1, 0, 0) + (0, 1, 1),$$

$$(1, 1, 1) = (1, 1, 0) + (0, 1, 1),$$

$$(1, 1, 1) = (1, 0, 0) + (1, 1, 0) + (0, 1, 1)$$

and so there are three corresponding sections $\sigma_1, \sigma_2, \sigma_3$. Let S be the two-element semilattice $\{1, e\}$ and let S act on M by having 1 act as the identity and e act via the map sending all non-zero elements to $(1, 1, 1)$ (and of course preserving zero). Then M is a $\mathbb{B}S$ -module. Let us compute $\chi_M(e)$. It is easy to compute $\chi_{M_{\sigma_1}}(e) = 2$, $\chi_{M_{\sigma_2}}(e) = 2$ and $\chi_{M_{\sigma_3}}(e) = 3$. Thus $\chi_M(e) = 2$. Notice that σ_3 corresponds to a redundant decomposition and hence has to be eliminated when taking the minimum.

Notice that if we were working over \mathbb{C} , then B would be a basis for \mathbb{C} and if e took each basis vector to $(1, 1, 1)$, then since $(1, 1, 1) = (1, 0, 0) + (0, 1, 1)$ we would also get 2 as being the character value on e .

Next we discuss the notion of a generalized character. Let M be a finite $\mathbb{B}S$ -module and $\min M \subseteq D \subseteq M \setminus \{0\}$. Then we define the *generalized character* $\psi_{M,D} : S \rightarrow \mathbb{N}$ by

$$\psi_{M,D}(s) = |\{m \in D \mid ms = m\}|.$$

For example, if S acts on the finite set X , then the generalized character $\psi_{\mathbb{B}X,X}$ is the complex character of the $\mathbb{C}S$ -module $\mathbb{C}X$ and also coincides with the min character introduced above. Let us define the *generalized character spectrum* $\text{cspec}(M)$ to be the set of all $\min M \subseteq D \subseteq M \setminus \{0\}$ such that $\psi_{M,D}$ is a complex character of S .

Proposition 4.14. *Let M be a non-zero finite $\mathbb{B}S$ -module. Let $\min M \subseteq D \subseteq M \setminus \{0\}$ be a subset such that $d \in D$ implies $ds \in D \cup \{0\}$ for all $s \in S$. Then $D \in \text{cspec}(M)$. In particular, $M \setminus \{0\} \in \text{cspec}(M)$.*

Proof. Let us write for the moment θ for the zero of M and put $D' = D \cup \{\theta\}$. Then S acts by total functions on D' . Thus $\mathbb{C}D'$ is a finite dimensional $\mathbb{C}S$ -module in a natural way and $\mathbb{C}\theta$ is a $\mathbb{C}S$ -submodule. Consider the $\mathbb{C}S$ -module $V = \mathbb{C}D'/\mathbb{C}\theta$. It is easy to see that $\psi_{M,D}$ is precisely the character of V . This completes the proof. \square

Recall that if S is a semigroup and R is an \mathcal{R} -class the Schützenberger representation of S on R is the actions of S on R by partial transformations given by $r \cdot s = rs$ if $rs \in R$ and otherwise is undefined (where $r \in R$ and $s \in S$). One can turn $\mathbb{C}R$ into a $\mathbb{C}S$ -module as follows. First let S act on $R \cup \{\square\}$ by sending all undefined products to \square and demanding $\square s = \square$ for all $s \in S$. Then consider the $\mathbb{C}S$ -module $\mathbb{C}(R \cup \{\square\})/\mathbb{C}\square$ (which as a vector space is isomorphic to $\mathbb{C}R$).

Theorem 4.15. *Let S be a finite semigroup and M_J an irreducible $\mathbb{B}S$ -module where we retain the notation of Theorem 4.4. Then there is subset $\min M \subseteq D \subseteq M \setminus \{0\}$ such that the generalized character $\psi_{M,D}$ is the complex character of S obtained by lifting the right Schützenberger representation of $\text{AGGM}_J(S)$ on an \mathcal{R} -class of its distinguished \mathcal{J} -class via the projection $\Gamma_J : S \rightarrow \text{AGGM}_J(S)$.*

Proof. Without loss of generality, we may assume that the action of S on M_J is faithful and hence $S = \text{AGGM}_J(S)$. In this case, since J is aperiodic, we may identify the action of S on B by partial transformations with the Schützenberger representation. Let $[b]$ denote the equivalence class of $b \in B$ under the congruence \equiv on \mathbb{B} from Theorem 4.4. Then $D = \{[b] \mid b \in B\}$ satisfies the conditions of Proposition 4.14. If we can show that the map $b \mapsto [b]$ is injective, then the proof of Proposition 4.14 will imply that $\psi_{M_J,D}$ is the complex character of the Schützenberger representation. Suppose that $[b] = [b']$ with $b \neq b'$. Let a be an \mathcal{R} -class of J and choose $s \in a \cap b$ and $s' \in a \cap b'$. Then, for $x \in B$, one has $[xs] = [b]$ if $C_{xa} = 1$ and is otherwise 0 and $[xs'] = [b']$ if $C_{xa} = 1$ and is otherwise 0. Thus s and s' act the same on M_J and so the action of S on M_J is not faithful, a contradiction. This completes the proof. \square

5. Density

This section relates our work with that of Zumbrägel [50]. In particular, we look at irreducible modules for semirings and discuss an application to semigroups. Of course, any irreducible module for a semiring R is an irreducible module for its underlying multiplicative semigroup, but the converse is false. So in principle if R is a finite semiring, then we can use our results to understand its irreducible representations as a semiring. One just has to determine which irreducible representations preserve the additive structure. However, since we are not assuming in general that the semirings in question are finite and also because proofs can become shorter and coordinate-free by taking advantage of the additive structure, we do not treat the representation theory of semirings as a special case of the representation theory of semigroups.

A semiring R is called *primitive* if it has a faithful irreducible module M . Let D be a division ring and M a left vector space over D . Then a subring $R \subseteq \text{End}_D(M)$ is said to be *dense* if, for any pair of

k -tuples (m_1, \dots, m_k) and (m'_1, \dots, m'_k) of linearly independent elements of M , there is an element $r \in R$ with $(m_1 r, \dots, m_k r) = (m'_1, \dots, m'_k)$. If R is a primitive ring with faithful irreducible module M , then $\text{End}_R(M)$ is a division ring D by Schur's lemma and so $R \subseteq \text{End}_D(M)$. Jacobson's density theorem shows that R must be a dense subring. Zornbrägel [50] proved an analogous result for finite primitive semirings with idempotent addition, although he stated it only under the hypothesis that R is congruence-simple. In fact, for finite semirings with idempotent addition, being primitive implies being congruence-simple. For finite rings, primitivity is also equivalent to simplicity since a finite primitive ring R will have a faithful finite irreducible module M and hence D above will be a finite division ring and thus a field by a theorem of Wedderburn. But then R will be a finite dimensional algebra over a field with a faithful irreducible module and hence by Wedderburn–Artin theory is simple.

Because Zornbrägel does not state his result [50] in full generality, we reproduce it for the reader's convenience. In what follows we assume that M is a join semilattice with minimum and maximum. For example any finite join semilattice with minimum also has a maximum. We shall always denote the maximum by ∞ . Following Zornbrägel [50], given $a, b \in M$, define

$$xe_{a,b} = \begin{cases} 0, & x \leq a, \\ b, & \text{else.} \end{cases}$$

One can check that the $e_{a,b} \in \text{End}(M)$ and that $\{e_{a,b} \mid a, b \in M\} \cup \{0\}$ is a subsemigroup E since

$$e_{a,b}e_{c,d} = \begin{cases} 0, & b \leq c, \\ e_{a,d}, & b \not\leq c. \end{cases}$$

It is shown below that M is an irreducible module for E . Zornbrägel defines $R \subseteq \text{End}(M)$ to be *dense* if it contains all the $e_{a,b}$ with $a, b \in M$. The next proposition does not require the semilattice to have a maximum.

Proposition 5.1. *Let M be a join semilattice with minimum and let E be as above. Then M is an irreducible $\mathbb{B}E$ -module. Thus any dense subsemiring of $\text{End}(M)$ is primitive.*

Proof. First observe that M is minimal. Indeed, if $0 \neq m, n \in M$ then $me_{0,n} = n$. Next suppose that \equiv is a non-trivial $\mathbb{B}E$ -module congruence on M . To show that \equiv is not proper, suppose $m \equiv n$ with $m \neq n$. Then without loss of generality we may assume $m \not\leq n$. Let $a \in M$. Then $me_{m,a} = 0$ and $me_{n,a} = a$. Thus $0 \equiv a$, completing the proof since a was arbitrary. \square

The following is [50, Theorem 2.3].

Theorem 5.2 (Zornbrägel). *Let M be a join semilattice with minimum and maximum and let $R \subseteq \text{End}(M)$ be dense. Then R is congruence-simple.*

The next result is a strengthening of the statement of [50, Proposition 3.13], but the proof is the same. The reader should compare with Theorem 4.4.

Theorem 5.3. *Let R be an idempotent semiring and M a faithful finite irreducible R -module. Then R is a dense subring of $\text{End}(M)$.*

Proof. Define a congruence on M by $m \equiv n$ if $mr = 0 \Leftrightarrow nr = 0$ for all $r \in R$. This is easily verified to be a congruence. Moreover, it cannot be universal since $m \equiv 0$ implies $mR = 0$. Thus it is the trivial congruence. Let I_m denote the annihilator in R of m . It is a right ideal.

First note that $e_{\infty,b} = 0$ and so belongs to R . So fix $\infty \neq a \in M$. We first show $e_{a,\infty} \in R$. Suppose $x \not\leq a$. We claim I_a does not annihilate x . Indeed, if I_a annihilates x , then since $(x+a)r = 0$ if and only

if $xr = 0 = ar$, it follows that $x + a \equiv a$ and so $x \leq x + a = a$. Now xI_a is then a non-zero submodule of M and so $xI_a = M$. Thus we can find $r_x \in I_a$ so that $xr_x = \infty$. Let

$$s = \sum_{x \not\leq a} r_x \in I_a.$$

Then if $x \leq a$, one has $xs \leq as = 0$, whereas if $x \not\leq a$, then $xs \geq xr_x = \infty$ and so $s = e_{a,\infty}$.

Now let $\infty \neq b \in M$. Then by irreducibility, $\infty R = M$ and so $b = \infty r$ for some $r \in R$. Then, $e_{a,\infty}r = e_{a,\infty}r = e_{a,b}$ and so R is dense, as required. \square

As a consequence, we obtain the following result of [50] (wherein the equivalence with the first item is not stated explicitly).

Corollary 5.4 (Zumbrägel). *Let R be a finite idempotent semiring with $R^2 \neq 0$. Then the following are equivalent:*

- (1) R is primitive;
- (2) R is isomorphic to a dense subsemiring of $\text{End}(M)$ for some finite semilattice M with minimum;
- (3) R is congruence-free.

Proof. The implication (2) implies (3) is [50, Theorem 2.3], whereas (3) implies (1) is [50, Proposition 3.10]. The final implication follows from Theorem 5.3 and the observation that any irreducible module for a finite idempotent semiring must be finite and idempotent (since $M = mR$ for any non-zero element $m \in M$). \square

It follows that if S is a finite semigroup, k is a finite idempotent semiring and M is an irreducible kS -module, then the span of the image of the representation $\varphi : S \rightarrow \text{End}_k(M)$ is dense. Zumbrägel showed [50, Proposition 4.9, Remark 4.10] that if M is finite, then $\text{End}(M)$ contains no proper dense subsemirings if and only if M is a distributive lattice. In particular, this applies when M is a finitely generated free \mathbb{B} -module since then M is isomorphic to the power set of its basis. We summarize this discussion in the next corollary.

Corollary 5.5. *Suppose S is a finite semigroup and let $\varphi : S \rightarrow \text{End}(M)$ be an irreducible representation where M is a join semilattice with identity. Then $\varphi(S)$ spans a dense subsemiring of $\text{End}(M)$. In particular, if M is a distributive lattice, then $\varphi(S)$ spans $\text{End}(M)$. Consequently, given an irreducible representation $\varphi : S \rightarrow M_n(\mathbb{B})$, the image of φ spans $M_n(\mathbb{B})$.*

The last part of the corollary can also be deduced in a straightforward way from Theorem 4.4. The key step is to note that the span of the rows of C is a free \mathbb{B} -module if and only if C has an identity submatrix of the appropriate rank.

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